

Interplay of disorder and interaction in Majorana quantum wires

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We study the interplay between disorder and interaction in one-dimensional topological superconductors which carry localized Majorana zero-energy states. Using Abelian bosonization and the perturbative renormalization group (RG) approach, we obtain the RG-flow and the associated scaling dimensions of the parameters and identify the critical points of the low-energy theory. We predict a quantum phase transition from a topological superconducting phase to a non-topological localized phase, and obtain the phase boundary between these two phases as a function of the electron-electron interaction and the disorder strength in the nanowire. We also identify a large regime of stability of the topological phase in the parameter space of the model.

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Introduction. The search for topological phases of matter has become an active and exciting pursuit in condensed matter physics [1]. Among the many important examples of such phases are topological superconductors supporting zero-energy Majorana bound states (MBS) [2–5]. A particularly promising realization of topological superconductivity is one-dimensional (1D) semiconductor/superconductor heterostructures [5]. In addition to being one of the simplest examples of fractionalization, zero-energy MBS quasiparticles have Ising-like non-Abelian braiding properties [6] and can be used for topological quantum computation [7].

The distinct feature of topological superconductors is the ground-state degeneracy due to the fermion parity encoded in the exponentially localized zero-energy MBS [4]. In a finite-length 1D wire, this degeneracy is approximate and there is an exponentially small energy splitting $e^{-L/\xi}$ due to a finite overlap of MBS [8]. Here L and ξ are the length of the wire (spatially separating the two MBS localized at the edges) and superconducting coherence length, respectively. The presence of the impurities in the wire adversely affects the stability of the topological phase by introducing disorder. Indeed, each impurity in a topological p-wave superconductor leads to the emergence of a localized subgap Andreev bound state. If the concentration of such Andreev states is too large, the overlap between MBS changes qualitatively from exponential to algebraic. Thus, increasing the disorder strength (e.g., by increasing the random impurity concentration) should lead to a topological quantum phase transition (QPT) from the Majorana-carrying topological superconducting phase with quantum degeneracy to a trivial phase with no end-MBS in the wire [9, 10]. In addition, the question of electron-electron interaction in the superconducting wire may have important implications for the topological phase, and there may be QPTs associated with the tuning of the interaction strength. It is well known that the properties of 1D conductor are

strongly affected by both electron-electron interactions and disorder [11]. Clarification of their combined effect is crucial for our complete understanding of the topological phase diagram of the system and ultimately for the experimental realization of Majorana quantum wires in the laboratory, where obviously both disorder and interactions would be ever-present.

In this Letter, we investigate an important question concerning the effect of both disorder and interaction on the stability of the topological phase and go beyond the non-interacting results of Refs.[9, 10, 12–16], where disorder effects on the topological superconducting phase were considered only in the non-interacting model, and of Refs. [17–20], where the effects of interaction have been studied in clean nanowires. We carry out a complete theoretical analysis in the presence of both disorder and interaction obtaining in the process the quantum topological phase diagram of the 1D nanowire system. We consider a semiconductor nanowire with strong spin-orbit and Zeeman interactions proximity-coupled to an s-wave superconductor (SC). As shown in Ref. [5], the low-energy Hamiltonian of the model in the suitable parameter regime reduces to the one of an effective 1D spinless p-wave SC. We then include the effects of both quenched disorder and interaction and, using Abelian bosonization and the replica method, derive a set of coupled renormalization-group (RG) equations for the parameters of the model. In general, disorder and repulsive interactions reinforce their detrimental effects on the topological SC phase and tend to eliminate the exponentially-split ground state MBS degeneracy associated with different fermion parity [21]. However, for a sufficiently strong initial induced pairing Δ , we predict a stable topological phase at low temperatures, even in the presence of disorder and interaction. Our results shed light on the question of the stability of MBS in realistic situations and are important for the ongoing experimental investigation searching for the MBS in semiconductor nanowire systems.

Theoretical model. We start with a model for a single channel conductor of length L with open boundary conditions, subject to Rashba SO coupling and a magnetic field along the wire (creating the Zeeman spin splitting). It is instructive to discuss first the non-interacting theory for the clean system. In that case, the Hamiltonian for the quantum wire in the continuum is [5]

$$H_0^{(1)} = \int dx \Psi^\dagger \left[-\frac{\partial_x^2}{2m} - \mu + i\alpha\sigma^y\partial_x - V_z\sigma^z \right] \Psi,$$

where $\hbar = 1$, $\Psi = (\psi_\uparrow(x), \psi_\downarrow(x))^T$ is the electron spinor operator, α is the Rashba SO interaction parameter, σ^i are the Pauli spin 1/2 matrices and V_z is the Zeeman splitting. The Hamiltonian $H_0^{(1)}$ is diagonalized in the Fourier basis and has eigenvalues $\epsilon_\pm(k) = k^2/2m - \mu \pm \sqrt{(\alpha k)^2 + V_z^2}$, with k the quasimomentum, and has the eigenmodes $\Psi_\pm(k)$ defining the higher and lower dispersion branches. The proximity-induced pairing interaction reads

$$H_0^{(2)} = \frac{i\Delta_S}{2} \int dx \Psi^\dagger \sigma^y \Psi^\dagger + \text{H.c.} \quad (1)$$

where Δ_S is the proximity-induced s -wave gap. Our approach consists in expressing the pairing term in the $\Psi_\pm(k)$ basis, which leads to SC pairing interactions with singlet (i.e., between Ψ_+ and Ψ_- subbands) and triplet (i.e., within the same Ψ_+ or Ψ_- subband) symmetry. The non-trivial topological phase supporting MBS occurs when the condition $V_z^2 > \Delta_S^2 + \mu^2$ is fulfilled [3], which physically implies that only the lowest subband Ψ_- is occupied. In that case, the system is an effective realization of the topological phase in the form of 1D spinless electrons with p-wave pairing [4].

At low energies we can linearize the band $\epsilon_-(k)$ around the Fermi points $\pm k_F$, and express the fermion field $\Psi_-(x)$ as [11]

$$\Psi_- = \frac{1}{\sqrt{2\pi a}} \left[U_R e^{ik_F x} e^{-i(\phi-\theta)} + U_L e^{-ik_F x} e^{i(\phi+\theta)} \right],$$

where the bosonic fields $\phi(x), \theta(x)$ are conjugate canonical variables obeying the commutation relation $[\phi(x), \theta(y)] = i\pi \text{sign}(y-x)/2$. Physically, $\phi(x)$ represents slowly-varying fluctuations in the electronic density $\rho(x) = \rho_0 - \partial_x \phi(x)/\pi$, while $\theta(x)$ is related to the SC order parameter. The Klein factors U_i keep track of the fermionic anticommutation relations and obey $\{U_i, U_j^\dagger\} = 0$ for $i \neq j$ and $U_i U_i^\dagger = U_i^\dagger U_i = 1$. Finally, $a \sim k_F^{-1}$ is the short-distance cutoff of the continuum theory. The bosonized Hamiltonian $H_0 = H_0^{(1)} + H_0^{(2)}$ is given by

$$H_0 = \int dx \left[\frac{vK}{2\pi} (\partial_x \theta)^2 + \frac{v}{2\pi K} (\partial_x \phi)^2 + \frac{2\Delta}{\pi a} \sin(2\theta) \right]. \quad (2)$$

Here Δ is the effective topological SC gap parameter, and for $\Delta = 0$ model (2) reduces to the Luttinger liquid (LL) model [11], which describes gapless plasmon excitations in the wire propagating with velocity $v \simeq v_F$, and is parametrized by the dimensionless Luttinger parameter $K < 1$ ($K > 1$) for repulsive (attractive) interactions. At large Δ , the field $\theta(x)$ is pinned to the minima of $\sin 2\theta$ and, thus, the superconducting state breaks $\mathbb{U}(1)$ symmetry down to \mathbb{Z}_2 . Indeed, in the limit $L \rightarrow \infty$, there are two degenerate minima $\theta(x) = -\pi/4, 3\pi/4$, related to each other by the global \mathbb{Z}_2 transformation $\theta \rightarrow \theta + \pi$, see Ref. [21] for details. Such a transformation is implemented by the fermion parity operator $P = (-1)^{N_F} = \exp \left[-i \int_0^L \partial_x \phi(x) dx \right]$ (where N_F is the total number of fermions), with even and odd degenerate eigenfunctions $|e, o\rangle = (|-\pi/4\rangle \pm |3\pi/4\rangle)/\sqrt{2}$ forming a fixed-fermion parity basis [21]. In Eq. (2) we have neglected the umklapp scattering which would introduce an additional term $\sim \cos(2\phi - 4k_F x)$ since we assume a filling incommensurate with the lattice. We are also taking the interaction to be short-ranged consistent with the canonical LL model, assuming strong screening in the nanowire (both by electrons in the semiconductor and by surrounding gate electrodes). The generalization to a true long-ranged unscreened Coulomb interaction does not change any of our conclusions as it only involves a very slowly-varying scale-dependent Luttinger parameter instead of a constant K [22].

We now introduce quenched disorder into model (2). The Hamiltonian for short-range non-magnetic impurities reads $H_i = - \int dx V_i(x) \Psi^\dagger(x) \Psi(x)$ (in the original notation), with Gaussian-distributed disorder potential $V_i(x)$ and $\langle V_i(x) V_i(y) \rangle = D_b \delta(x-y)$. In addition, we consider that imperfections at the semiconductor-superconductor interface gives rise to an inhomogeneous tunnel-coupling mechanism producing the SC proximity-effect in the nanowire [15]. In that case, second-order perturbation theory in the tunnel-coupling generates, in addition to the uniform contribution (1), a spatially fluctuating s-wave pairing, i.e., $H_{\text{dis}, S} = \frac{i}{2} \int dx \delta\Delta_S(x) \Psi^\dagger(x) \sigma^y \Psi^\dagger(x) + \text{H.c.}$, where we assume $\delta\Delta_S(x)$ to be a Gaussian variable obeying $\langle \delta\Delta_S(x) \delta\Delta_S(y) \rangle = D_s \delta(x-y)$. Although in principle $\delta\Delta_S(x)$ is a complex quantity encoding both amplitude and phase fluctuations, the latter can be neglected in the case of a proximate 3D SC at temperatures $T \ll \omega_p$, where $\omega_p = \sqrt{4\pi\rho_{3D}e^2/m^*}$ is the 3D plasma frequency. Upon projection onto the lowest subband Ψ_- , we obtain the bosonized Hamiltonians

$$H_i = \int dx \left[-\eta(x) \frac{\nabla \phi(x)}{2\pi} + \xi(x) \frac{e^{-i2\phi(x)}}{2\pi a} + \text{H.c.} \right], \quad (3)$$

$$H_{\text{dis}, \Delta} = \frac{1}{\pi a} \int dx \zeta(x) \sin 2\theta(x). \quad (4)$$

Here we have defined the disordered potentials $\eta(x) \equiv \frac{1}{N} \sum_{q \sim 0} e^{iqx} V_i(q)$, $\xi(x) \equiv \frac{1}{N} \sum_{q \sim 0} e^{iqx} V_i(q - 2k_F)$ and $\zeta(x) \equiv -\frac{1}{N} \sum_{q \sim 0} \cos(qx) \delta\Delta_S(q) \alpha / 2V_z$. The forward scattering term $-\eta(x) \nabla \phi(x) / 2\pi$ can be eliminated by means of a gauge transformation $\phi(x) \rightarrow \phi(x) - \frac{K}{v} \int^x dy \eta(y)$, reflecting the fact that forward scattering does not affect the thermodynamic properties of the system [11, 23]. We next implement the replica method, that consists in introducing the set of “replicas” of the system $\phi(x), \theta(x) \rightarrow \phi_i(x), \theta_i(x)$, with $i = 1, 2, \dots, n$, allowing a simpler integration over different disorder configurations [11, 24]. After integrating out the Gaussian fields V_i and $\delta\Delta_S$, the replicated action becomes

$$S = \sum_{i,j=1}^n \left[\delta_{ij} S_{0,i} - \frac{D_b}{(2\pi a)^2} \int_{x,\tau,\tau'} \cos 2[\phi_i(x,\tau) - \phi_j(x,\tau')] \right. \\ \left. + \frac{D_s}{(\pi a)^2} \sum_{\nu=\pm 1} (\nu) \int_{x,\tau,\tau'} \cos 2[\theta_i(x,\tau) + \nu \theta_j(x,\tau')] \right], \quad (5)$$

$$S_{0,i} = -i \int d\tau dx \frac{1}{\pi} \nabla \theta_i(x,\tau) \partial_\tau \phi_i(x,\tau) + \int d\tau H_{0,i}(\tau).$$

where the Hamiltonian $H_{0,i}$ is defined in Eq. (2). In the absence of SC pairing, this model has been studied in the context of the localization transition, predicted to occur at the critical value $K_c = 3/2$, in the limit of weak disorder and spinless fermions [23]. For $K < K_c$, disorder flows to strong coupling and the groundstate corresponds to a pinned charge-density-wave (PCDW), characterized by a localization length $\xi_{\text{loc}} \propto D_b^{1/(3-2K)}$. Above K_c , the LL phase remains stable, describing a “delocalized” electronic fluid.

RG analysis. The critical properties of model (5) can be studied in the framework of perturbative RG around the Luttinger liquid fixed-point. Following standard derivations [11], we expand the partition function corresponding to action S at first-order in the small parameters D_b and D_s , and up to second order in Δ . We implement a RG procedure that leaves invariant the Gaussian sector of the action and obtain the following system of RG-flow equations

$$\frac{dK(\ell)}{d\ell} = \frac{2}{\pi} \left[\frac{A_3}{K(\ell)} y_\Delta^2(\ell) + \frac{4A_2}{K(\ell)} y_s(\ell) - K^3(\ell) B_2 y_b(\ell) \right], \quad (6)$$

$$\frac{dv(\ell)}{d\ell} = -\frac{2}{\pi} \left[B_2 K^2(\ell) y_b(\ell) + \frac{4A_2}{K^2(\ell)} y_s(\ell) \right] v(\ell), \quad (7)$$

$$\frac{dy_\Delta(\ell)}{d\ell} = \left(2 - \frac{1}{K(\ell)} \right) y_\Delta(\ell), \quad (8)$$

$$\frac{dy_b(\ell)}{d\ell} = (3 - 2K(\ell)) y_b(\ell), \quad (9)$$

$$\frac{dy_s(\ell)}{d\ell} = \left(3 - \frac{2}{K(\ell)} \right) y_s(\ell). \quad (10)$$

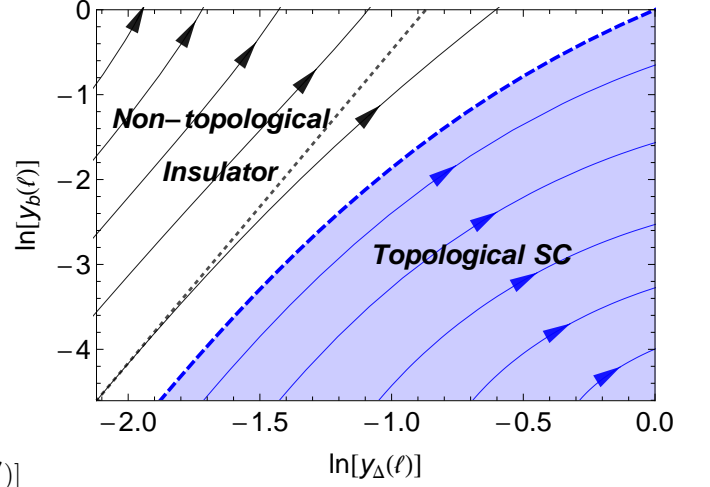


Figure 1. Parametric dependence of $y_b(\ell)$ vs $y_\Delta(\ell)$, as obtained from the numerical solution of the RG-flow Eqs. (6)-(10), for fixed initial parameters $K_0 = 0.65$ and $y_{s0} = 0$ (log-log scale). The thick dashed curve is the critical line, separating the topological SC phase (shaded area) from the non-topological disordered phase, and the thin dotted line is our analytical estimate $y_b \sim y_\Delta^2$, valid in the limit $\{y_b(\ell), y_\Delta(\ell)\} \rightarrow 0$.

Here we have introduced the dimensionless variables $y_\Delta = \Delta / v\Lambda$, $y_b = D_b / \Lambda v^2$, $y_s = D_s / \Lambda v^2$ (with $\Lambda \sim k_F$ the high-energy cutoff of the theory), and the non-universal numerical coefficients $A_{2,3}$ and B_2 of order unity. Physically Eq. (6) describes the renormalization of interactions in the wire (parametrized by $K(\ell)$) induced by superconductivity and disorder. While $y_\Delta(\ell)$ and $y_s(\ell)$ couple to field $\theta(x)$, and favor a SC ground state with \mathbb{Z}_2 -symmetry, the parameter $y_b(\ell)$ couples to the dual field $\phi(x)$ and tries to pin the density to the disorder potential, thus opposing a SC ground state. These competing effects are reflected in the different signs of the prefactors in Eq. (6): $y_\Delta(\ell)$ and $y_s(\ell)$ renormalize $K(\ell)$ to larger values, inducing attractive interactions in the wire, while $y_b(\ell)$ drives $K(\ell) \rightarrow 0$ enhancing the effect of repulsive interactions. We therefore see that repulsive interaction and disorder reinforce each other’s detrimental effects on the topological SC. In the limit $\{y_\Delta(\ell), y_b(\ell), y_s(\ell)\} \rightarrow 0$, the properties of the system are determined by the value of $K(\ell)$, i.e., the couplings $y_\Delta(\ell)$ and $y_s(\ell)$ become relevant for $K(\ell) > 1/2$ and $K(\ell) > 2/3$, respectively whereas $y_b(\ell)$ is relevant for $K(\ell) < 3/2$ [23].

In order to simplify the analysis, we focus first on the case $y_s = 0$. In that case, note that within the experimentally interesting regime $1/2 < K(\ell) < 3/2$ both $y_\Delta(\ell)$ and $y_b(\ell)$ are competing perturbations flowing simultaneously to strong coupling. Moreover, in the non-interacting case $K = 1$, $y_\Delta(\ell)$ and $y_b(\ell)$ have the same scaling dimension. In order to maintain the in-

ternal consistency of our perturbative approach, the RG flow has to be stopped at a value ℓ_{\max} for which one of the couplings reaches the strong-coupling regime, i.e., $\max[y_{\Delta}(\ell_{\max}), y_b(\ell_{\max})] = 1$. Although strictly speaking our approach is not applicable in the strong-coupling regime, the fact that $\theta(x)$ and $\phi(x)$ are dual fields that cannot order simultaneously allows us to reasonably conjecture that there are no intermediate fixed-points in the RG flow, and therefore to classify the nature of the ground state according to the coupling that first reaches the above condition. When the two competing couplings reach the strong coupling regime simultaneously (i.e., $y_b(\ell_{\max}) = y_{\Delta}(\ell_{\max}) = 1$), the system does not order and this condition defines a critical line of QPTs that separates the topological SC phase with broken \mathbb{Z}_2 symmetry from the PCDW insulating phase (cf. dashed line in Fig. 1).

From the lowest order RG equations one obtains $y_b(\ell) = y_{b0}e^{(3-2K)\ell}$, $y_{\Delta}(\ell) = y_{\Delta0}e^{(2-K^{-1})\ell}$, which together produces the relative scaling $y_b \sim y_{\Delta}^{\nu}$ with $\nu = (3 - 2K) / (2 - K^{-1})$. Physically, this means that interactions (encoded in ν) determine the scaling of disorder strength relative to the SC order parameter: for $K > 1$ (attractive interactions) disorder grows slower than SC, while the inverse occurs for $K < 1$ (repulsive interactions). In Fig. 1 we show the parametric dependence of $y_b(\ell)$ as a function of $y_{\Delta}(\ell)$, for initial parameters $K_0 = 0.65$ and $y_{s0} = 0$. The continuous lines correspond to the numerical solution of Eqs. (6)-(10), and the dotted line is our analytical result $y_b \sim y_{\Delta}^{\nu}$, valid in the limit $\{y_b(\ell), y_{\Delta}(\ell)\} \rightarrow 0$. At the phase boundary (thick dashed line in Fig. 1), this result implies the approximate relation $y_{b0} = y_{\Delta0}^{\nu}$ for the initial values, which together with the relation between impurity scattering time and $D_b: 1/\tau_e = 2n_i u_0^2 \pi \mathcal{N}_0 = 2D_b/v_F$ (here n_i is the concentration of impurities, u_0 the typical impurity potential and $\mathcal{N}_0 = 1/\pi v_F$ the 1D density of states), produces $1/2\tau_e E_F = (\Delta/E_F)^{\nu}$. Interestingly, for $K = 1$ we find that the critical condition for the topological-non-topological transition is $1/2\tau_e = \Delta$, which exactly coincides with the results obtained in the non-interacting case [9, 10, 12–14, 16].

The results shown in Fig. 1 demonstrate that a topological superconducting state that supports MBS could be in principle realized in a realistic semiconducting quantum wire, subject to the simultaneous effects of disorder and repulsive interaction, provided it is in the proper regime in parameter space.

The above procedure leads to a qualitative “phase-diagram” in terms of the initial parameters of the model. In Fig. 2 we plot the critical curves in $y_{\Delta0}$ - y_{b0} space, for different initial values of interaction K_0 . The area below each curve represents the regime for which topological SC is expected to dominate over disorder, and therefore stable MBS to exist. Starting from the initial

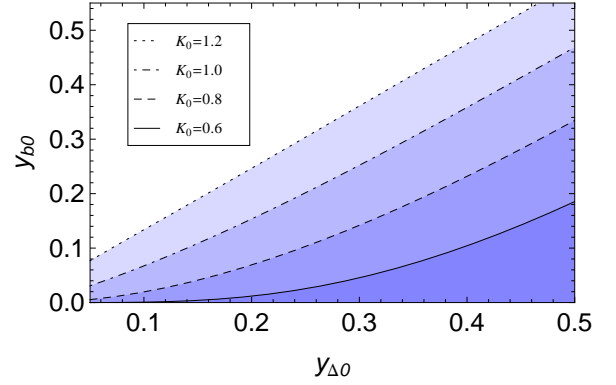


Figure 2. Phase diagram in $y_{\Delta0}, y_{b0}$ space obtained for $y_{s0} = 0$ and different values of K_0 . The curves correspond to the critical lines y_{b0} vs $y_{\Delta0}$, satisfying the condition $y_{\Delta}(\ell_{\max}) = y_b(\ell_{\max}) = 1$. The area below each curve represents the regime for which topological SC is expected to dominate over disorder.

value $K_0 = 0.65$, representing a strongly interacting wire, note that the topological region expands as the interaction becomes increasingly attractive.

As mentioned before, for $K > 3/2$ and in the absence of pairing $\Delta = 0$, disorder is irrelevant and the ground state of the wire corresponds to a stable LL [23]. In our case, we see from Eq. (8) that the LL fixed point is *unstable* against a vanishingly small pairing interaction y_{Δ} , and directly flows to the stable topological phase [17]. This result generalizes the localization transition in the non-SC problem to the case of topological SC.

We now turn to the effects of a non-vanishing y_s . The lowest order RG-flow Eq. (6) suggests a favorable effect on the topological SC ground state through a positive renormalization of $K(\ell)$. Note however that this conclusion is only valid in the limit of small fluctuations $y_s(\ell) \ll y_{\Delta}(\ell)$ (i.e., $|\delta\Delta_S(x)| \ll \Delta_S$). Actually, recent numerical results indicate that for typical fluctuations $|\delta\Delta_S(x)|$ larger than $\sim \Delta_S/4$, the topological state is destroyed [15]. In our description, the adverse effects on the topological SC are encoded in the sign-changes of the random potential $\zeta(x)$, which induce spatial fluctuations in $\theta(x)$. From the RG flow Eq. (10) we extract a scaling dimension $3 - 2/K(\ell)$ for the coupling $y_s(\ell)$, which for $K(\ell) < 1$ is smaller than that of $y_{\Delta}(\ell)$. Therefore, in our regime of interest $K(\ell) < 1$, and for initial values $y_{s0} \ll y_{\Delta0}$, we expect disorder in the interface tunneling-amplitude to be a perturbative correction to the average pairing term $\sim \Delta \sin(2\theta)$, and our results of Eq. (6) to be valid. A more general study of disorder in the interface tunneling-amplitude will be presented elsewhere [25].

Conclusions. We have carried out an RG analysis of the topological superconductivity in 1D Majorana chain systems, arising in semiconductor nanowires due to the combined effects of spin-orbit coupling, Zeeman splitting,

and s-wave proximate superconductivity, in the presence of interaction and disorder treating them on equal footing. Our main conclusion is that the topological phase with non-Abelian Majorana bound states in the chain is stable in a large regime of the parameter space.

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- [1] F. Wilczek, *Nature Physics* **5**, 614 (2009).
 - [2] N. Read and D. Green, *Phys. Rev. B* **61**, 10267 (2000), S. Das Sarma, C. Nayak, and S. Tewari, *Phys. Rev. B* **73**, 220502 (2006), L. Fu and C. L. Kane, *Phys. Rev. Lett.* **100**, 096407 (2008), C. Zhang *et al.*, *Phys. Rev. Lett.* **101**, 160401 (2008), M. Sato and S. Fujimoto, *Phys. Rev. B* **79**, 094504 (2009), J. Alicea, *Phys. Rev. B* **81**, 125318 (2010).
 - [3] J. D. Sau *et al.*, *Phys. Rev. Lett.* **104**, 040502 (2010).
 - [4] A. Kitaev, *Usp. Fiz. Nauk* **171**, 131 (2001).
 - [5] R. M. Lutchyn, J. D. Sau, and S. Das Sarma, *Phys. Rev. Lett.* **105**, 077001 (2010), Y. Oreg, G. Refael, and F. von Oppen, *Phys. Rev. Lett.* **105**, 177002 (2010).
 - [6] D. A. Ivanov, *Phys. Rev. Lett.* **86**, 268 (2001).
 - [7] C. Nayak *et al.*, *Rev. Mod. Phys.* **80**, 1083 (2008).
 - [8] M. Cheng *et al.*, *Phys. Rev. Lett.* **103**, 107001 (2009).
 - [9] O. Motrunich, K. Damle, and D. A. Huse, *Phys. Rev. B* **63**, 224204 (2001).
 - [10] I. A. Gruzberg, N. Read, and S. Vishveshwara, *Phys. Rev. B* **71**, 245124 (2005).
 - [11] T. Giamarchi, *Quantum Physics in One Dimension* (Oxford University Press, Oxford, 2004).
 - [12] P. W. Brouwer *et al.*, *Phys. Rev. Lett.* **85**, 1064 (2000).
 - [13] P. W. Brouwer *et al.*, *Phys. Rev. B* **84**, 144526 (2011).
 - [14] P. W. Brouwer *et al.*, *Phys. Rev. Lett.* **107**, 196804 (2011).
 - [15] T. D. Stanescu, R. M. Lutchyn, and S. Das Sarma, *Phys. Rev. B* **84**, 144522 (2011).
 - [16] J. D. Sau, S. Tewari, and S. Das Sarma (2011), arXiv:cond-mat/1111.2054, *Phys. Rev. B* (in press).
 - [17] S. Gangadharaiah *et al.*, *Phys. Rev. Lett.* **107**, 036801 (2011).
 - [18] E. Sela, A. Altland, and A. Rosch, *Phys. Rev. B* **84**, 085114 (2011).
 - [19] R. M. Lutchyn and M. P. A. Fisher, *Phys. Rev. B* **84**, 214528 (2011).
 - [20] E. M. Stoudenmire *et al.*, *Phys. Rev. B* **84**, 014503 (2011).
 - [21] L. Fidkowski *et al.*, *Phys. Rev. B* **84**, 195436 (2011).
 - [22] H. J. Schulz, *Phys. Rev. Lett.* **71**, 1864 (1993). D. W. Wang, A. J. Millis, and S. Das Sarma, *Phys. Rev. B* **64**, 193307 (2001).
 - [23] T. Giamarchi and H. J. Schulz, *Phys. Rev. B* **37**, 325 (1988).
 - [24] S. F. Edwards and P. W. Anderson, *J. Phys. F* **5**, 965 (1975).
 - [25] A. M. Lobos, R. Lutchyn, and S. Das Sarma, in preparation.